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The Transition from Formula-Centered to Concept-Centered Analysis Bolzano's Purely Analytic Proof as a Case Study

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Résumé : Deux transitions ont eu lieu aux XVIII^e et XIX^e siècles dans le développement de l'analyse mathématique : de l'approche géométrique à l'approche axée sur des formules d'une part ; de l'approche axée sur les formules à l'approche conceptuelle d'autre part. En nous appuyant sur la Preuve purement analytique de Bolzano, nous montrons qu'il adopte une approche que l'on peut qualifier de conceptuelle. Nous parvenons à la conclusion selon laquelle Bolzano n'adopte pas la même attitude selon qu'il se rapporte à l'approche géométrique d'une part, à l'approche axée sur des formules d'autre part ; dans le premier cas, il est question de rejet, dans le second cas de non-participation. Bolzano appuie sa méthodologie conceptuelle sur des opinions philosophiques partagées en partie par certains mathématiciens partisans d'une approche de l'analyse axée sur les formules.

Abstract: In the 18^{th} and 19^{th} centuries two transitions took place in the development of mathematical analysis: a shift from the geometric approach to the formula-centered approach, followed by a shift from the formula-centered approach to the concept-centered approach. We identify, on the basis of Bolzano's *Purely Analytic Proof* [Bolzano 1817], the ways in which Bolzano's approach can be said to be concept-centered. Moreover, we conclude that Bolzano's attitude towards the geometric approach on the one hand and the

formula-centered approach on the other were of a different nature; the former being one of rejection, the latter of non-participation. Bolzano supports his concept-centered methodology by philosophical views, which were partially shared by mathematicians with a formula-centered approach to analysis.

1 Introduction

Bernard Bolzano's (1781–1848) treatise "*Rein analytischer Beweis*"[1] ([Bolzano 1817]; henceforth [*RAB*]) has been the subject of many studies. Typically these studies discuss the relation between [*RAB*] and Cauchy's work—did Cauchy plagiarise Bolzano, or not [Grattan-Guiness 1970b], [Freudenthal 1971], [Benis Sinaceur 1973], [Grabiner 1984], [Bottazzini 1986, 97–98]?—focus on the mathematical content of [*RAB*] (see [Grattan-Guiness 1970a, 51–57, 71–75]; [Bottazzini 1986, 99–101]; [Rusnock 2000, 73–84]) and point out Bolzano's advanced standards of rigor of proofs (see [Bottazzini 1986, 98]) and their underlying philosophical ideas (see [Rusnock 2000, 69–73]). It is usually emphasized that Bolzano was a precursor of later developments in mathematics and that his work thus did not fit into the common mathematical practice of his time. Bottazzini writes, for example, that

> [t]he arguments that Bolzano brought to his demonstrations and the motives that he brought to his method of reasoning were completely unusual in the context of mathematics at the time. [Bottazzini 1986, 97]

In the current paper we emphasize another side of the story, one touched upon only marginally in the studies just mentioned: we show how [*RAB*] reflects the transitions in the development of mathematics of Bolzano's time, especially transition from a so-called *formula-centered* to a *concept-centered* approach, i.e from an approach in which mathematics is viewed as primarily concerned with formulas and their manipulation to an approach in which concepts, conceived of as independent of any particular formal representation, take center stage.[2]

We argue that [*RAB*] not only presented Bolzano's remarkable ideas on proofs—ideas both absent in 18^{th} century mathematical practice and ultimately rooted in a millenia old Aristotelian conception of deductive sciences (cf. [Betti & de Jong 2010][3])—but also that [*RAB*] clearly and in many places

1. In full: "*Rein analytischer Beweis des Lehrsatzes, daß zwischen je zwey Werthen, die ein entgegensetztes Resultat gewaehren, wenigstens eine reelle Wurzel der Gleichung liege*".

2. This terminology goes back to Sørensen, cf. [Sørensen 2005, 454ff] for a more detailed account.

3. Betti and de Jong call this the *Classical Model of Science*.

witnesses the important transitions of mathematics of the early 19th century. By discussing [*RAB*] from the perspective of two major transitions in analysis of the 18th and early 19th century, we contribute to a more balanced account of Bolzano's [*RAB*]. This paper may be considered as a detailed elaboration focused on [*RAB*] of Rusnock's observation:[4]

> We find him [Bolzano] at the forefront of the movement to recast the calculus as real analysis, moving from the geometrical and algebraic understanding of the subject common in the eighteenth century to one based on logical and arithmetical concepts. [Rusnock 2000, 56][5]

Let us briefly sketch which changes took place in analysis in the 18th and 19th century. Two major transitions can be identified: First a change from analysis based on geometrical conceptions to an algebraic approach starting in the 1740s, and second a change from that algebraic approach to an arithmetical approach at the beginning of the 19th century [Fraser 1989, 317]; [Lützen 2003, 156]. As we will discuss below, the algebraic approach is formula-centered, whereas the arithmetical one is concept-centered.

A proponent of the first transition was Euler, who as early as 1740 rejected a geometrical proof of a certain theorem of differential calculus because such a proof, he argued, would be "drawn from an alien source" (see [Fraser 1989, 319]). We will see that Bolzano uses a similar argument to reject a proof in [*RAB*]. Instead, Euler's approach was *formula-centered* [Sørensen 2005].

According to the formula-centered approach mathematics can be said to deal primarily with analytic *expressions* (formulas) and algebraic manipulations (or: calculations) of these expressions:

> A function was usually regarded as being an analytic expression; it might be a polynomial, a rational function or an explicit algebraic function; it might involve logarithms, exponentials or trigonometric functions. It might also involve series or products or continued fractions, and it was assumed that the rules of formal algebra applied to these irrespective of any considerations of convergence. [Smithies 1986, 42]

Although there was no universal consensus on the notion of function in the 18th century, the view that a function is an analytic expression can be found,

4. A view opposite to Rusnock's can be found in [Kitcher 1975], who puts [*RAB*] in the algebraic tradition and attacks the view that Bolzano inaugurated the arithmetisation of analysis (see esp. [Kitcher 1975, 244]). Kitcher's main argument is that a certain mistake in Bolzano's proof can only be explained from an algebraical point of view, and we do not deny that also certain algebraic traits can still be found in [*RAB*]. For further critique on [Kitcher 1975], see [Johnson 1977, 264, fn 2].

5. Similar, more extensive remarks can be found in [Russ 2004, 141–147]. His observations, however, are not concentrated especially on [*RAB*]. Neither does he emphasize the transition from the formula-centered to the concept-centered approach as we will in this paper.

e.g., in early Euler [Lützen 2003, 156]; [Euler 1748]. A wider notion of function, also employed by Euler, allows a function to be given on different intervals by different analytic expressions [Fraser 1989, 326]. A notion of continuity that fits squarely within the formula-centered approach considers a function (in the wider sense) to be continuous if and only if it is given throughout by a *single* analytic expression [Smithies 1986, 43]; [Jahnke 2003, 124]. Note that this definition of continuity does not capture the property of being connected: $\frac{1}{x}$ is a continuous function according to it, even though the function has a vertical asymptote at $x = 0$.

The assumption that the rules of formal algebra are applicable to analysis has been baptised "the principle of the generality of algebra" (following Cauchy's terminology). Fraser characterises this algebraic approach to analysis as follows:

> The algebraic calculus studies functional relations, algorithms and operations on variables. The values that these variables receive, their arithmetic or geometric interpretation, are of secondary concern. [...] The calculus of EULER and LAGRANGE differs from later analysis in its assumptions about mathematical existence. The relation of this calculus to geometry or arithmetic is one of correspondence rather than representation. [Fraser 1989, 328]

Particularly illustrative in this context is the specific interpretation of equality within the formula-centered approach, sometimes called *formal equality*, which is a conception of equality governed by formal, algebraic rules. According to this conception, an equality does not express a relation between numbers, but rather states that certain *expressions* are equivalent in a certain sense—independently of whether they can be interpreted numerically. "Theorems" were hence often seen as nothing more than rules, which could hold even if numerical exceptions were known or if a numerical interpretation was lacking altogether, as in certain infinite series or in parts of infinitesimal calculus.

With the decline of the formula-centered approach, the emphasis shifted towards an interpretation of equality as a relation strictly between numbers. This was witnessed by broad methodological change: If one works with a notion of numerical equality rather than formal equality, and if theorems of analysis are not interpreted as rules but as truths about a certain domain of quantities, then numerical exceptions can no longer be admitted and a new quest for rigor of proofs begins.

This shift from formal equality to numerical equality is exemplary for the transition that took place, leading away from the central role of expressions. Instead, concepts took that role, often generalising certain traits that were earlier expressed as properties of these expressions [Lützen 2003, 165]; [Grabiner 1984, 113]. For example, while Euler's notion of continuity, mentioned above, was defined as being given by a single analytical expression, during the shift to the conceptual approach to analysis a gradual consensus arose that the notion of continuity should be independent of any particular representation of functions.

In what follows we will show that Bolzano's mathematical practice, and especially [*RAB*], can be understood in terms of these historical shifts. We will see that he rejected the geometrical approach just as his formula-centered precursors did. Moreover, we will maintain that Bolzano's concept-centered tendency appeared not so much in explicit criticisms he made of formula-centered proofs—as was the case with the geometrical approach—but was rather implied in his practice. We will further note that Bolzano does not categorically oppose the methods that are associated with the formula-centered approach. He approvingly mentions, for instance, certain proofs by Gauss that were purely algebraic [*RAB*, 253]. Moreover, we will see that Bolzano argued in favor of the new rigor of proofs by referring to the Aristotelian ideal of a science.

In order to do this we will begin by giving an overview of Bolzano's distinctively Aristotelian take on the sciences in general and on mathematics in particular as it is presented in his early *Beyträge zu einer begründeteren Darstellung der Mathematik* (henceforth [*BD*]).[6]

2 Philosophical background

Bolzano's [*RAB*] is often seen as a paradigm example of how philosophical considerations can influence mathematical practice in a fruitful way. It has been pointed out that Bolzano's take on what a proper scientific proof should accomplish led him to strive for a rigour that enabled him to achieve original results in analysis (cf. [Mancosu 1996, 93], [Mancosu 1999], [Detlefsen 2008], [Rusnock 2000]). Central to the relevant philosophical considerations of Bolzano is the idea that all true propositions stand in a unique objective explanatory order, which in later works he calls *grounding* (*Abfolge*). We will call this order also the *grounding order* or the *order of grounds and consequences.* In [*BD*] he gives the following description of this order:

> [T]his much seems to me certain: in the realm of truth, i.e., the collection of all true judgements, a certain *objective connection* prevails, which is independent of our accidental *subjective recognition* of it. As a consequence of this some of these judgements are the grounds of others and the latter are the consequences of the former. Presenting this objective connection of judgements and placing them one after another so that a consequence is repre-

6. We will not go much into the details of this treatise, and refer the interested reader to the treatments in [Rusnock 2000, chap. 2] and [Centrone 2012].

> sented as such and conversely, seems to me to be the real *purpose* to pursue in a scientific exposition. [*BD*, 39–40][7]

Regardless of the basis on which we actually come to know a given truth, it occupies a place in a certain objective order. Bolzano argues that proper *scientific proofs* (or, in Bolzano's later terminology, *demonstrations* (*Begründungen*)) should follow this order of grounds and consequences:

> [B]y a *scientific proof* of a truth [we understand] the representation of the *objective dependence* of it on *other truths*, i.e., the derivation of it from those truths which must be considered as *the ground for it*—not fortuitously, but *actually* [an sich] *and necessary*—while the truth it self, in contrast, must be considered as their consequence. [*BD*, 64]; cf. also [Bolzano 1837, § 525], henceforth [WL]

Bolzano conceived of scientific proofs or demonstrations as *explanatory*. The grounds on which a given proposition depends *explain why* that proposition is true [*WL*, § 177] (cf. [Mancosu 1999]). That a proof is explanatory, though, does not mean that it is the most convincing one. And conversely, that a proof is convincing does not mean necessarily that it is explanatory. In particular, thus, that a theorem is obviously true does not deprive the investigator from searching which place it occupies in the explanatory order, i.e., from searching for the grounds that reveal *why* it is true.

Bolzano was never able to clarify exactly what this objective order looks like in a manner that fully satisfied him. In his early work—which stands in the background of [*RAB*]—Bolzano introduces a number of conditions that proper scientific presentations and proofs have to meet from which we can derive some fundamental properties of the order.[8] First of all, proofs that explain why a certain theorem holds must not rely on that very theorem (or on other truths that rely on it) in the course of the proof. Secondly, there is at most one correct such proof for any truth in the order [*BD*, II § 30]. Finally, Bolzano argues that proper scientific proofs should proceed from general to specific propositions, and from simple to complex propositions [*BD* II, §§ 26,27] and that they should be *pure*, i.e., not "cross to another kind" ([*BD* II, § 29]

7. Page references are to the German original [Bolzano 1810a]. English translations are mostly taken from [Bolzano 1810b]. In some cases we depart somewhat from the translation.

8. In his later work *Wissenschaftslehre* ([*WL*]), Bolzano gave a much more detailed account of the grounding relation [*WL*, § 198–221], though we will not need to go into this for the purposes of this paper. The interested reader may consult [Tatzel 2002], [Buhl 1961], the relevant sections of [Sebestik 2011], and [Betti 2010]. Neither the idea that the realm of truths is ordered by an explanatory relation, nor the specific properties Bolzano ascribes to the order were novel inventions by him (as he acknowledges, cf. [*BD*, II § 26]). The first chapter of [Mancosu 1996] offers a useful overview of related concerns by predecessors of Bolzano.

and [Bolzano 1804], second of unnumbered pages).[9] These last ideas can be nicely illustrated by Bolzano's classification of the mathematical disciplines put forward in [*BD*], (cf. [*BD* I, § 20]).

A. General mathematics (deals with "things in general")

B. Special mathematical disciplines (relative to "special things" they deal with)

   I. Aetiology (deals with the theory of causes and effects of "unfree things")

   II. Theory of "unfree sensible [i.e., perceivable] things"...

      a. ... according to their form "*in abstracto*"

         α. Theory of time (where time is the respective form)

         β. Theory of space (where space is the respective form)

      b. ... according to their form "*in concreto*"

         α. Temporal aetiology

         β. Pure natural science

We can see a certain *order* with respect to the generality of the respective disciplines within the classification. The *domain* of general mathematics consists of things "in general", whereas the domain of, e.g., *pure natural science* consists of only those things that are concrete, perceivable and situated in space. We can also note that in each of the disciplines certain *notions* are introduced by which the domain of the respective discipline is narrowed down (e.g., the notion of *time* and the notion of *concrete object* cf. [Rusnock 2000, 33] and the literature cited therein). As we shall see, the theorem that Bolzano sets out to prove in [*RAB*] belongs to general mathematics. Notably, Bolzano's general mathematics does not encompass geometry. Rather, the latter constitutes a branch of special mathematics (which Bolzano in other contexts also calls *applied mathematics*), namely the theory of space.

Bolzano's views on grounding imply that within proper scientific proofs truths that belong to disciplines that are higher in the hierarchy must not be proven by reliance on truths that belong to disciplines lower in the order (though it is allowed to prove truths lower in the order by truths higher in it). Doing so would break the prohibition on "crossing to another kind". Note, however, that also internal to the disciplines truths are ordered with respect to their generality and complexity. Using a complex and specific truth in a proof of a more simple and more general one from the same discipline is thus also inadmissible.

Bolzano views mathematical truths as being part of an order that is independent of the human mind and also independent of any particular representation. Proper scientific proofs are supposed to determine the place that

9. The question of how these conditions are precisely to be understood goes beyond the scope of this paper. Cf. [Centrone 2012] for a thorough discussion.

a given theorem occupies in that order. Since this place is determined by the specific concepts of which the truth is composed, as well as its complexity and extension, acquiring a grasp of the concepts contained in a theorem becomes an indispensable precondition for rigorous proofs [Rusnock 2000, 59]. One must find correct definitions of the concepts that occur in a given theorem and determine the *domain* of objects to which it applies rather than 'blindly' manipulate a formal representation of the theorem.

3 *Purely analytic proof*

Bolzano's [RAB] appeared in 1817, in the early days of the transition from the formula-centered to the concept-centered approach. The full title of [RAB] reads: "Purely analytic proof of the theorem that between any two values, which give results of opposite sign, there lies at least one real root of the equation". In order to illustrate the transitions visible in Bolzano's paper, it is instructive to take a close look at how Bolzano phrases (and paraphrases) this theorem precisely, and how he sets out to prove it.

The theorem mentioned in the title of [RAB] refers to polynomial equations with rational coefficients (cf. [Rusnock 2000, 69]). It is stated in full explicitly in the last section of [RAB] as follows:

> If a function of the form
>
> $$x^n + ax^{n-1} + x^{n-2} + \cdots + px + q$$
>
> in which n denotes a positive integer, takes a *positive* value for $x = \alpha$, but a *negative* value for $x = \beta$, then the equation
>
> $$x^n + ax^{n-1} + x^{n-2} + \cdots + px + q = 0$$
>
> has at least one *real root* lying between α and β. [RAB, 276]

Let's call this theorem the "Opposite Sign Theorem" (OST). A special case of OST had been used without proof by Gauss to prove the Fundamental Theorem of Algebra (henceforth FTA) in [Gauss 1815] and [Gauss 1816], which Bolzano praises as a proof that can be understood "in a purely analytical sense" [RAB, 253]. By this, Bolzano as well as Gauss mean—*negatively*—that the proof does *not cross to another kind* in making use of geometrical considerations (cf. [Gauss 1815, 33]; [Russ 2004, 144]). We shall see, though, that even though Bolzano and Gauss agree to a certain extent on what a purely analytic proof must *not* make use of, their opinions differ as to which methods are preferable in such a proof. While Gauss's methods in his "new" proofs of FTA can be placed squarely in a formula-centered approach, Bolzano's proof of OST clearly is a step towards a concept-centered perspective.

This becomes evident from the way Bolzano phrases the theorem in the foreword of [RAB] in the context of discussing other proofs for OST. Even though OST (as stated in the title of [*RAB*]) is a theorem about the roots of polynomial equations, Bolzano frequently refers to the theorem "which is to be proved" in a different way in the course of the foreword. He paraphrases it, for example, as follows:

> [(I)] (...) every continuous function of x which is positive for one value of x, and negative for another, must be zero for some intermediate value of x. [*RAB*, 255]

And in a slightly different context "the very proposition which we wish to establish" is stated thus:

> [(II)] (...) every continuous variable function of x, which is positive for $x = \alpha$, and negative for $x = \beta$, must be zero for some value between α and β. [*RAB*, 258]

Since Bolzano does not mention any other theorem in the foreword apart from FTA, it seems that (I) and (II) are intended as paraphrases of OST. But in those paraphrases the theorem does not appear any longer as a claim about polynomial equations, but instead as a claim about continuous functions and their values. Note that the notion of continuity, which we find in (I) and (II), neither occurs in the title of Bolzano's treatise, nor in the fully explicit statement of OST in [*RAB*, § 18] quoted above.

In approaching a problem that stemmed from a formula-centered approach to mathematics, Bolzano thus immediately provides a reinterpretation from the point of view of a concept-centered approach. This shift in perspective necessitated a proof quite different than those given for OST beforehand, which Bolzano carefully surveyed. In particular, a rigorous definition of the notion of continuity that appears in (I) and (II) turned out to be necessary for the proof. We will discuss Bolzano's survey of other proofs of OST in the next section. But let us first offer a few remarks on the course of Bolzano's own proof. It is not difficult to see that paraphrases (I) and (II) are in their wording very close to the Intermediate Value Theorem (IVT). The latter theorem is central for Bolzano's proof of OST (cf. [*RAB*, § 15]) and it is this theorem for which Bolzano's paper is nowadays most famous. This theorem is stated as follows:

> If two functions of x, fx and ϕx, vary *according to the law of continuity* either for all values x or for all those lying between α and β, and furthermore if $f\alpha < \phi\alpha$ and $f\beta > \phi\beta$, then there is always a certain value of x between α and β for which $fx = \phi x$. [*RAB*, 273]

Bolzano's proof of this theorem has been discussed extensively in the literature.[10] We will content ourselves with quickly pointing out how Bolzano shows

10. Cf. [Rusnock 2000, chap. 3.3], [Russ 2004, 148–151], and the literature cited therein. Rusnock and Russ also provide a discussion of one crucial flaw that can be found in Bolzano's proof.

that OST can be proved by means of IVT. He proceeds as follows. First, he shows that "[e]very function of the form $a + bx^m + cx^n + \ldots + px^r$, in which $m, n, \ldots, r$ denote positive integer exponents, is a quantity which varies *according to the law of continuity*" [*RAB*, 275]. In other words, he shows that the polynomials involved in the equations that OST is about are (or determine) continuous functions [*RAB*, § 17]. Subsequently, he makes use of IVT to show that these functions will have the value zero for some x (where $\alpha < x < \beta$) in case their value is positive for α and negative for β, and argues that this value is then the root of the corresponding polynomial equation [*RAB*, § 18]. Bolzano's proof thus reduces a theorem which deals of functions "of a certain form" to a more general one concerning continuous functions, which is independent of the particular representation of the functions.

4 Rejections of other proofs

Bolzano saw the original contribution of his proof, as sketched in the previous section, not in the presumed fact that he had shown the theorem to be true. This would have been superflous given its general acceptance. Rather, he saw his contribution as having provided a *demonstration*—a proof that situates the theorem in the objective order of grounds and consequences.

To promote his proof, Bolzano therefore also discussed other, known proofs of OST, and explained why they were not acceptable as demonstrations. We discuss three of these rejections below, two of which concern proofs that mathematicians with a formula-centered approach would also (or might) reject. The fact that they rejected the same proofs, sometimes even for similar reasons, does not mean that Bolzano followed their mathematical practice. In one of Bolzano's rejections, for example, he argued for an arithmetical definition of continuity of functions that fits in the concept-centered approach, rather than in the formula-centered. So, although there seems to have been some consensus in rejecting geometrical proofs for theorems from analysis, there was no agreement concerning what they should be replaced by.

The last rejection that we discuss is one of a proof by Lagrange with a markedly formula-centered approach. The reason that Bolzano gave for rejecting this proof did not directly relate to issues surrounding the formula-centered approach. Rather, he supported his rejection on the basis of his ideas concerning the grounding relation, and specifically on the claim that the ground of a given truth never lies in a more complex truth.

4.1 Proof depending on a geometrical truth

The first proof discussed and subsequently rejected by Bolzano is one that depends on the following geometrical truth:

> Every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely), must necessarily intersect with the abscissae-line somewhere at a point lying between those ordinates. [*RAB*, 254]

This proposition is, according to Bolzano, correct and obvious, but using it to derive OST is not a *demonstration* (see Section 2). In a proper scientific proof of a truth from general mathematics such as OST, one must not appeal to truths which belong to a more specific discipline (geometry):

> (...) the strictly scientific proof, or the objective ground of a truth, which holds equally for *all* quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in *space*. [*RAB*, 254]

Bolzano refers to the prohibition of kind crossing when rejecting this geometrical proof of a truth from general mathematics [*RAB*, 254], just as, for example Euler and also Gauss had done in similar cases, as mentioned above in the Introduction and Section 3. Furthermore, Bolzano explains that kind crossing (in the current proof) leads to a circularity, if the proof would be a proper demonstration:

> If we adhere to this view [that the objective reason of a truth which holds equally for *all* quantities can lie in a truth which holds merely for quantities which are in *space*] we see instead that such a *geometrical* proof is, in this as in most cases, really circular. [*RAB*, 254]

His reasoning goes as follows. First, he argues that the geometrical truth cannot possibly be an *axiom*. Hence, there will be truths that constitute its ground. Second, he claims that the most plausible candidate for being a ground for the geometrical truth in question is the "general truth" OST. Since the grounding relation is non-circular according to Bolzano, this proof of OST from the geometrical truth is rejected.

In other words: Bolzano links the prohibition of kind crossing to his theory of grounding, yet this is not the only way in which his rejection distinguishes itself from similar rejections by mathematicians who had a formula-centered approach. The main difference is not situated in the reasons for rejecting this geometric proof, but rather in the fact that Bolzano takes a concept-centered approach to OST, as shown by his reformulation (I) of this theorem (see previous section), which appears in this context.

4.2 Proof using concepts of time and motion

The second proof that Bolzano rejects on the basis of methodological considerations reads as follows:[11]

11. Note that this proof is actually not directly a proof of OST, but of IVT, which is also part of general mathematics in Bolzano's sense.

> 'If two functions fx and ϕx', they say, 'vary according to the law of continuity and if for $x = \alpha$, $f\alpha < \phi\alpha$, but for $x = \beta$, $f\beta > \phi\beta$, then there must be some value u, lying between α and β, for which $fu = \phi u$. For if we imagine that the variable quantity x in both these functions successively takes all values between α and β, and in both always takes the same value at the same moment, then at the *beginning* of this continuous change in the value of x, $fx < \phi x$, and at the *end*, $fx > \phi x$. But since both functions, by virtue of their continuity, must first through all intermediate states, there must be some *intermediate moment* at which they were both equal to one another.' [*RAB*, 255]

Clearly, temporal vocabulary is employed in the statement of the proof ("at the same moment", "the beginning"). The proof, Bolzano continues, is then further illustrated by the example of the motion of two bodies. As notions of *time* and *motion* are "alien to general mathematics" [*RAB*, 255], by an argument analogous to the one given above, a proof that makes use of these notions cannot count as a demonstration. However, Bolzano does not reject this proof on *that* basis. He points out that *expressions* for time and space are used in a non-essential way, "just to avoid the constant repetition of the same word" [*RAB*, 255–256]. Accordingly Bolzano rephrases the proof without temporal or spatial expressions. It is exactly in his reformulation—which in the end he also rejects for methodological reasons—that we can identify his concept-centered approach.

For example, the notion of continuity that Bolzano distills from the proof is the following [*RAB*, 256]:

> A function fx varies according to the law of continuity for all values of x inside or outside certain limits, when $f(x + n\Delta x)$ can take every value between fx and $f(x+\Delta x)$ if n is taken arbitrarily between 0 and 1.

Although Bolzano argues that the above is not the correct definition of continuity, but rather a theorem that is actually a special case of IVT, it is telling that he extracts the above definition (instead of one in formula-centered terms). Even more significant is his own proposal for a definition of continuity:

> According to a *correct definition*, the expression *that a function fx varies according to the law of continuity for all values of x inside or outside certain limits* means only that, *if x is any such value the difference $f(x + \omega) - fx$ can be made smaller than any given quantity, provided ω can be taken as small as we please.* [*RAB*, 256]

On first sight the above definition does not refer to analytic expressions and can thus be regarded as concept-centered. The footnote Bolzano attaches to his definition of continuity makes this impression even stronger:

> There are functions which are continuously variable for *all* values of their argument [*Wurzel*], e.g., $\alpha + \beta x$. However, there are also others which vary according to the law of continuity only for values of their argument inside or outside certain limits. Thus $x + \sqrt{(1-x)(2-x)}$ varies continuously only for all values of x which are $< +1$ or $> +2$, but not for the values which lie between $+1$ and $+2$. [*RAB*, 256]

Bolzano mentions several analytic expressions in this footnote, but only in order to counter formula-centered definitions like Euler's, in which, as mentioned, a function is continuous if and only if it is given by a single analytic expression. The function of the last example, $x + \sqrt{(1-x)(2-x)}$, is given by a single analytic expression and would thus be considered continuous *tout court* on Euler's definition. Bolzano, introducing a local, numerical notion of continuity that captures the geometric intuition of being connected, must instead specify the domain on which he considers the function to be continuous, leaving out the interval on which the function has no (real) value.

On the other hand, it can also be argued that the footnote shows that in [*RAB*] Bolzano was still partially rooted in the formula-centered approach. First, the examples in the footnote are single analytic expressions. This means it is not only unclear whether Bolzano is here considering at all functions not given by analytic expressions, but it is even unclear whether he accepts functions given on different intervals by different analytic expressions.[12]

Second, the word that Bolzano uses for "argument", *Wurzel* ("*root*"), also bears a clear reference to functions as (analytic) expressions. It is, in fact, a value of an unknown which satisfies an equation (see [Russ 2004, 256, fn. f]), such as in the title of [*RAB*].

What we can conclude from Bolzano's definition is that he introduced a local, numerical notion of continuity that did not mention analytic expressions and that was thus in principle also broad enough to capture continuity for a wider notion of function.

4.3 Proof depending on decomposition into factors

Just before Bolzano wrote [*RAB*], Gauss had given two proofs of the fundamental theorem of algebra (FTA) in a formulation that avoids the use of complex numbers [Gauss 1815], [Gauss 1816]; see also [Cain 2005, 1]:

> *[E]very algebraic rational integral function of one variable quantity can be decomposed into real factors of first or second degree.* [*RAB*, 253]

12. The acceptance of functions given on different intervals by different analytic expressions as a solution to the vibrating string equation was the subject of the famous debate between Euler and D'Alembert (see [Fraser 1989, 326]). It is clear that, in his later mathematical works, Bolzano rejected the view that functions must be expressible by an analytic expression (cf. [Bolzano 1833-1841, 231f]).

According to Bolzano, Gauss's proofs of FTA "leave hardly anything to be desired" [*RAB*, 253]. Yet when subsequently Lagrange had derived OST from FTA [Lagrange 1808], Bolzano found this latter proof unacceptable.

The fact that Bolzano rejects Lagrange's proof is of particular interest, because of his apparent agreement with Gauss's aim to avoid *crossing to another kind* in his proof, i.e., to avoid any appeal to truths from geometry (see [*RAB*, 253]; [Gauss 1815, Sect. 1]). This means that of the rejections we discuss, this is the only one in which a proof is rejected that can be placed in the formula-centered approach.

Bolzano's reason for rejecting Lagrange's proof is twofold. First, he argues that a proof of OST from FTA is not acceptable as a *demonstration*:

> But the fact remains that such a derivation could not be called a *demonstration*, in that the second proposition [FTA] clearly expresses a much *more complex* truth than our present one [OST]. The second can therefore certainly be based on the first, but not, conversely, the first on the second. [*RAB*, 258]

Second, Bolzano points out that Gauss's proof of FTA, on which Lagrange's proof relies, makes tacit use of OST. So a proof of OST from FTA would actually be circular, and indeed a logical mistake.

Note that neither of the reasons Bolzano gives for rejecting Lagrange's proof has a direct relation to the formula-centered approach. Bolzano does not criticise Lagrange for making a wrong application of algebraic rules or holding an incorrect notion of continuity. Rather, the former is based upon Bolzano's philosophical ideas and the latter on general logical considerations.

5 Conclusion

We have identified a number of elements within Bolzano's mathematical practice that show a clear tendency towards a concept-centered approach in [*RAB*]. This tendency is visible, in particular, in Bolzano's reformulations of OST and in his employment of a local, numerical definition of continuity. Motivated by his philosophical views on proper scientific proofs, Bolzano was trying to identify and define the concepts which occur in the main theorem of his paper. Accordingly, his proof draws on traits of those concepts rather than on traits of a particular formulation of the theorem. While Bolzano's concept-centered approach can be seen retrospectively as an advancement on the formula-centered one, his attitude towards the latter was one of non-participation rather than of overt criticism. This is witnessed by the fact that he claims Gauss's purely algebraic proofs of FTA leave "hardly anything to be desired" [*RAB*, 253]. This stands in stark contrast to his attitude towards the geometric approach, which he criticised more than once. It is in the rejection of the geometric approach that we even find a profound agreement between Bolzano's methodological views and the views of others more typically regarded as having favoured a formula-centered approach.

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